Mathematical Demonstration of the Continuity of Time and Space

Limits, Smoothness, and Finite Resolution

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Abstract

We present a rigorous mathematical demonstration of the continuity, derivability, and infinite smoothness of a function defined on an extended domain obtained by adjoining a single limit point at infinity. A simple but explicit construction is introduced in which finite indices represent successive stages of refinement, while the point at infinity encodes the limiting value of the process. It is shown that the function is continuous at infinity, that its derivative at infinity exists and vanishes, and that all higher-order derivatives are likewise well-defined and equal to zero. The limiting point is therefore not a singular boundary, but the smooth completion of a convergent sequence.

By mapping the behavior at infinity to a finite point through an appropriate change of variables, the analysis avoids informal notions of derivatives at infinity and establishes differentiability in a fully standard and rigorous sense. The resulting infinite smoothness demonstrates that the approach to the limit is asymptotically flat, with no residual slope, curvature, or higher-order obstruction.

From a physical standpoint, this result clarifies a common conceptual error: finite measurement resolution does not imply ontological discreteness. The existence of smallest measurable intervals reflects experimental limitation rather than a breakdown of continuity. The continuity of time and space emerges as a property of the limiting structure itself, stable under arbitrarily fine refinement and incompatible with the inference of intrinsic granularity from finite resolution alone.

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1.1 Definition of the Function

Let $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ denote the set of natural numbers extended by a single limit point ∞.

We define the function $y : \overline{\mathbb{N}} \to \mathbb{R}$ in piecewise form as

$$y(n) = \begin{cases} a + 1 - 10^{-n}, & \text{for } n < \infty \text{ (i.e., } n \in \mathbb{N}), \\ a + 1, & \text{for } n = \infty. \end{cases}$$

This definition makes explicit a distinction that is often left implicit in physical and mathematical reasoning.

Finite values of the index n correspond to **finite-resolution descriptions or approximations**. Each such value represents a concrete step in a refinement process, but none of them exhausts the structure being approximated.

The symbol $n = \infty$ does not represent an additional finite element of the domain. It denotes the **limit** of the finite approximations. Assigning a value to $y(\infty)$ introduces no new structure: the value $a + 1$ merely names the limit toward which the sequence $\{y(n)\}_{n \in \mathbb{N}}$ converges.

This type of extension is standard in real analysis. A convergent sequence can be canonically completed by adjoining its limit point, yielding a closed and well-defined object without altering the behavior of the sequence at finite indices.

Crucially, the function is not defined "at infinity" in a physical sense. Infinity here labels a **mode of convergence**, not a measurable state. The piecewise form serves to separate what is accessible through finite steps from what is defined by limiting behavior.

This separation will be essential in the subsequent analysis, where finite measurement resolution must be distinguished from the continuity of the underlying structure.

1.2 Preliminary Observation

For every finite index $n \in \mathbb{N}$, the function satisfies

$$y(n) \neq a + 1,$$

since the correction term 10^{-n} is strictly positive. Nevertheless, the deviation from the limiting value is given exactly by

$$|y(n) - (a + 1)| = 10^{-n}.$$

This simple expression already encodes the essential mathematical features of the construction.

First, for any finite n, the error 10^{-n} is strictly positive. Consequently, the value $a + 1$ is never attained by any finite element of the domain. This reflects the fact that finite approximations, by definition, do not exhaust the limiting structure.

Second, the sequence $\{y(n)\}_{n \in \mathbb{N}}$ converges monotonically. Indeed, as n increases, the quantity 10^{-n} decreases, and one finds

$$y(n + 1) - y(n) = (a + 1 - 10^{-(n+1)}) - (a + 1 - 10^{-n}) = 10^{-n} - 10^{-(n+1)} > 0.$$

The sequence is therefore strictly increasing.

Third, the sequence is bounded above. For all $n \in \mathbb{N}$,

$$y(n) = a + 1 - 10^{-n} < a + 1,$$

so $a + 1$ serves as a global upper bound.

Finally, there exists no strictly positive lower bound on the error term. Given any $\delta > 0$, one can always choose n large enough such that

$$10^{-n} < \delta.$$

Hence, there is no $\delta > 0$ for which $10^{-n} \geq \delta$ holds for all n. The only lower bound compatible with all finite indices is zero.

These observations capture the precise mathematical meaning of "approaching a value without reaching it": no finite step attains the limit, yet the deviation from the limit can be made arbitrarily small.

1.3 Limit at Infinity

We now evaluate the limit of $y(n)$ as the index n grows without bound. By direct computation,

$$\lim_{n \to \infty} y(n) = \lim_{n \to \infty} \left(a + 1 - 10^{-n} \right).$$

Since the exponential term satisfies

$$\lim_{n \to \infty} 10^{-n} = 0,$$

it follows immediately that

$$\lim_{n \to \infty} y(n) = a + 1.$$

By definition of the piecewise extension introduced earlier,

$$y(\infty) = a + 1.$$

Hence, the value assigned to the function at $n = \infty$ coincides exactly with the limit of the finite sequence.

This step contains no hidden assumptions. The limit exists because the sequence is monotone increasing and bounded above, which is a standard sufficient condition for convergence in real analysis. Importantly, the limit is determined solely by the asymptotic behavior of the finite terms; it does not depend on any notion of infinite precision or infinite measurement.

The role of the limit here is purely structural: it completes the sequence by identifying the value toward which all finite approximations converge.

1.4 Continuity at $n = \infty$

We now establish the continuity of the function y at the point $n = \infty$.

Since ∞ is not a finite element of $\mathbb{N}$, continuity at $n = \infty$ is defined through limiting behavior. The function y is said to be **continuous at infinity** if

$$\lim_{n \to \infty} y(n) = y(\infty).$$

From the previous section, we have already shown that

$$\lim_{n\to\infty} y(n) = a + 1,$$

and by definition of the piecewise extension,

$$y(\infty) = a + 1.$$

Therefore,

$$\lim_{n\to\infty} y(n) = y(\infty),$$

and the function is continuous at $n = \infty$.

This conclusion can also be expressed using the εN definition of continuity. Let $\varepsilon > 0$ be arbitrary. We must show that there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|y(n) - y(\infty)| < \varepsilon.$$

Using the explicit form of the function, we compute

$$|y(n) - y(\infty)| = |(a + 1 - 10^{-n}) - (a + 1)| = 10^{-n}.$$

The inequality $10^{-n} < \varepsilon$ is satisfied whenever

$$n > \log_{10}\left(\frac{1}{\varepsilon}\right).$$

Thus, choosing

$$N = \left\lceil \log_{10}\left(\frac{1}{\varepsilon}\right) \right\rceil + 1,$$

ensures that for all $n \geq N$,

$$|y(n) - y(\infty)| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the εN condition is satisfied, and the function y is continuous at $n = \infty$.

This result shows that the extension of the function by its limiting value introduces no discontinuity. The point $n = \infty$ represents the smooth closure of the finite sequence, not a singular boundary.

2 Derivability at $n = \infty$

Continuity at infinity establishes that the function approaches its limiting value without jumps. We now study a stronger property: **derivability at infinity**, which characterizes how smoothly the approach occurs.

A direct "difference quotient at infinity" is not meaningful. The standard rigorous method is to map the behavior at infinity to the behavior near a finite point via a change of variables.

2.1 Definition (Derivability at Infinity)

Let $y(n)$ be defined for large n. Define a new variable $t = 1/n$, so that $n \to \infty$ corresponds to $t \to 0^+$. Define the associated function

$$g(t) := y\left(\frac{1}{t}\right), \quad t > 0,$$

and set

$$g(0) := y(\infty).$$

We say that y is **derivable at** $n = \infty$ if g is derivable at $t = 0^+$. In that case, the derivative "at infinity" is defined as $g'(0)$.

2.2 Application to the Present Function

For $n < \infty$ (i.e., $n \in \mathbb{N}$),

$$y(n) = a + 1 - 10^{-n}, \qquad y(\infty) = a + 1.$$

Hence, for $t > 0$,

$$g(t) = y\left(\frac{1}{t}\right) = a + 1 - 10^{-1/t}.$$

Using $10^{-1/t} = e^{-(\ln 10)/t}$, this may be written as

$$g(t) = a + 1 - e^{-(\ln 10)/t}, \quad t > 0, \qquad g(0) = a + 1.$$

2.3 Computation of $g'(0)$

We compute the derivative at 0 using the definition:

$$g'(0) = \lim_{t \to 0^+} \frac{g(t) - g(0)}{t}.$$

Substituting $g(t)$ and $g(0)$,

$$\frac{g(t) - g(0)}{t} = \frac{(a + 1 - e^{-(\ln 10)/t}) - (a + 1)}{t} = -\frac{e^{-(\ln 10)/t}}{t}.$$

As $t \to 0^+$, the exponential term $e^{-(\ln 10)/t}$ tends to zero faster than any power of t. In particular,

$$\lim_{t \to 0^+} \frac{e^{-(\ln 10)/t}}{t} = 0,$$

and therefore

$$g'(0) = 0.$$

2.4 Conclusion

The derivative at infinity exists (in the standard, rigorous sense defined above), and it is equal to zero. Thus, $y(n)$ is **derivable at** $n = \infty$, and the approach to the limiting value is not only continuous but also asymptotically flat:

$$\boxed{y \text{ is derivable at } n = \infty, \text{ with derivative } 0.}$$

3 Higher-Order Derivability and Smoothness at $n = \infty$

We now study whether the function is not only once derivable at infinity, but **derivable to all orders**, i.e., whether the limiting point $n = \infty$ is reached with full smoothness.

As before, the correct notion of higher-order differentiability at infinity is defined through the change of variables $t = 1/n$. Define

$$g(t) = y\left(\frac{1}{t}\right), \quad t > 0, \qquad g(0) = y(\infty).$$

Then y is k-times derivable at $n = \infty$ if and only if g is k-times derivable at $t = 0^+$. If this holds for all $k \in \mathbb{N}$, then y is C^∞ at infinity.

For the present function,

$$y(n) = a + 1 - 10^{-n} \quad (n \in \mathbb{N}), \qquad y(\infty) = a + 1,$$

we have

$$g(t) = a + 1 - 10^{-1/t} = a + 1 - e^{-(\ln 10)/t}, \quad t > 0, \qquad g(0) = a + 1.$$

To analyze higher derivatives, note that for $t > 0$ every derivative of $e^{-(\ln 10)/t}$ can be written as the exponential itself multiplied by a rational function of t. More precisely, for each integer $k \geq 1$, there exists a polynomial P_k such that

$$\frac{d^k}{dt^k}\left(e^{-(\ln 10)/t}\right) = P_k\left(\frac{1}{t}\right) e^{-(\ln 10)/t}, \quad t > 0.$$

Consequently, for each $k \geq 1$,

$$g^{(k)}(t) = -\,P_k\left(\frac{1}{t}\right) e^{-(\ln 10)/t}, \quad t > 0.$$

The crucial asymptotic fact is that the exponential decay dominates any polynomial growth: for every integer $m \geq 0$,

$$\lim_{t \to 0^+} t^m\, e^{-(\ln 10)/t} = 0.$$

Equivalently, for any polynomial $Q(1/t)$,

$$\lim_{t \to 0^+} Q\left(\frac{1}{t}\right) e^{-(\ln 10)/t} = 0.$$

Applying this to $g^{(k)}(t)$, we obtain for every $k \geq 1$,

$$\lim_{t \to 0^+} g^{(k)}(t) = 0.$$

Thus we can consistently define

$$g^{(k)}(0) = 0, \qquad \forall k \geq 1.$$

It follows that g is C^∞ at $t = 0$, and hence the original function y is **infinitely differentiable at** $n = \infty$ in the rigorous sense induced by the transformation $t = 1/n$. Moreover, all derivatives at infinity vanish:

$$g^{(k)}(0) = 0 \quad \forall k \geq 1.$$

Therefore, the limit is approached not only continuously and with zero first derivative, but with **complete smoothness**: there is no residual slope, no curvature, and no higher-order obstruction at the limiting point.

4 Physical Meaning of Smoothness at $n = \infty$

The mathematical result established in the previous sections has a direct physical interpretation once the role of the refinement parameter n is made explicit.

Let n label the **effective resolution** with which a physical quantity is accessed. Increasing n corresponds to improving measurement precision, refining spatial or temporal discrimination, or reducing experimental uncertainty. The limit $n \to \infty$ does not describe a physical operation, but an idealization in which all finite limitations are removed.

Infinite smoothness at $n = \infty$ means that the corrections to the limiting value vanish faster than any inverse power of the refinement parameter. There is no residual slope, no curvature, and no higher-order obstruction as resolution increases. In physical terms, the underlying structure responds smoothly to arbitrarily fine probing.

This has a clear conceptual consequence. If time or space were fundamentally discrete, one would expect signatures of that discreteness to appear as the probing scale approaches the alleged fundamental unit: non-differentiable behavior, kinks, plateaus, or residual finite slopes. The absence of such features in the limiting behavior shows that intrinsic granularity cannot be inferred solely from limits of measurement resolution.

Reports of "minimum measurable" times or lengths therefore reflect the current state of experimental capability, not a structural endpoint of nature. Reducing the probing scale encounters increasing technical difficulty, but no mathematical or physical boundary at which continuity fails.

From a structural point of view, smoothness at infinity expresses **stability**. The continuum of time and space is not an approximation that breaks down under refinement; it is reinforced by refinement. Continuity is not merely preservedit is maximally stable, surviving unlimited improvement in resolution without generating singular behavior.

In this sense, infinite differentiability at the limit provides a precise mathematical formulation of a continuous ontology: the structure of time and space is defined by limits, not by the finite steps through which those limits are approached.

This result does not exclude discrete or lattice-based models of spacetime. Rather, it establishes that smooth limits and infinite differentiability of observables are compatible with arbitrarily fine refinement and therefore cannot, by themselves, be used as evidence for or against an underlying discrete ontology. The argument addresses the logical relation between measurement resolution and continuity, not the ultimate microscopic constitution of spacetime.

5 Conclusion

We have constructed a complete and internally consistent analysis of a simple function defined on an extended domain, showing that its behavior at infinity is continuous, derivable, and infinitely smooth.

Starting from a piecewise definition that explicitly separates finite indices from the limiting point, we showed that finite approximations never attain the limiting value, yet converge monotonically toward it. The limit exists, is uniquely defined, and coincides with the value assigned at infinity, establishing continuity without ambiguity.

By mapping the behavior at infinity to a finite point through a change of variables, we demonstrated that the function is not only continuous but also derivable at infinity, with a vanishing first derivative. Extending the analysis further, we proved that all higher-order derivatives exist and vanish at the limit, showing that the approach to the limiting value is infinitely smooth.

From a mathematical standpoint, the point at infinity is therefore not a singular boundary, nor a site of structural breakdown. It is the smooth completion of a convergent process defined entirely by finite terms.

From a physical standpoint, this result clarifies a common conceptual error: finite measurement resolution does not imply ontological discreteness. The existence of a smallest measurable interval does not entail the existence of a smallest physical unit. Continuity, differentiability, and smoothness are properties of the limiting structure, not of the finite steps used to approximate it.

In this sense, the continuity of time and space does not require infinite precision, infinite measurements, or additional hypotheses. It follows directly from the existence and smoothness of the limit, independently of any assumption about a fundamental minimal scale.

References

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